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2001 J. Phys. A: Math. Gen. 34 L567

(http://iopscience.iop.org/0305-4470/34/42/102)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) L567–L576 PII: S0305-4470(01)27077-2

LETTER TO THE EDITOR

From the braided to the usual Yang–Baxter relation

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Received 17 July 2001
Published 12 October 2001
Online at stacks.iop.org/JPhysA/34/L567

Abstract
Quantum monodromy matrices coming from a theory of two coupled (m)KdV equations are modified in order to satisfy the usual Yang–Baxter relation. As a consequence, a general connection between braided and *unbraided* (usual) Yang–Baxter algebras is derived and also analysed.

PACS numbers: 11.30.-j, 02.20.-a, 02.40.-k, 03.50.-z

1. Preliminaries

The Yang–Baxter relation for the monodromy matrix of a quantum system ([1,2] and references within) characterizes the integrability of the system. Indeed, this relation encodes a quantum group *symmetry* [3,4], i.e. an infinite-dimensional deformed Lie algebra including the Abelian conserved quantities—which enables the system to be *à la* Liouville integrable—as a Cartan subalgebra. In some interesting cases this very rich structure seems to be missing because of non-ultralocal commutation relations between the fundamental variables (or fields) of the theory. Nevertheless, a suitable modification of the Yang–Baxter relation was discovered to hold and to still ensure the Liouville integrability: the so-called braided Yang–Baxter relation [5–7]. However, this relation is substantially different from the usual Yang–Baxter relation and without obvious links to it.

Within the huge variety of integrable non-ultralocal theories the prototype is the very interesting case of the quantum (modified) KdV ((m)KdV) system, which gives [8–10] an alternative description of the minimal conformal field theories (CFTs) [11]. The associated lattice monodromy matrix verifies the braided version of the Yang–Baxter relation [6] and the transfer matrix can be diagonalized by means of a generalization of the algebraic Bethe ansatz method [7]. If a left theory of this kind is properly coupled to a right theory, the resulting monodromy matrix still satisfies the braided Yang–Baxter relation and was conjectured in [7] to give, on a cylinder, an alternative description of minimal CFTs perturbed by the $\Phi_{1,3}$ primary operator [12]. On the plane these theories are also described as suitable restrictions of the sine-Gordon theory [13], which, in contrast, exhibits a Yang–Baxter relation [1].

In this letter we will address the very natural question of the connection between the braided and the usual Yang–Baxter relation. We will use as examples the cases described in the previous paragraph and involving the lattice (m)KdV theory as the main ingredient. Actually, in what follows we will show that our treatment is completely general. We will conclude the letter by proposing a new *factorized* monodromy matrix verifying the Yang–Baxter relation.

In a previous paper [7] we considered two copies of the modified KdV equation:

$$\partial_\tau v = \tfrac{3}{2}v^2 v' + \tfrac{1}{4}v''' \qquad \partial_{\bar\tau}\bar v = \tfrac{3}{2}\bar v^2 \bar v' + \tfrac{1}{4}\bar v''' \tag{1.1}$$

where $v \equiv -\varphi'$ and $\bar v \equiv -\bar\varphi'$ are the spatial derivatives of quasi-periodic Darboux fields defined on the interval $y, \bar y \in [0, R]$. The quantizations of the Darboux fields are the Feigin–Fuks bosons [14] which satisfy the commutation relations

$$[\phi(y), \phi(y')] = -\frac{\mathrm{i}\pi\beta^2}{2}s\left(\frac{y-y'}{R}\right) \qquad [\bar\phi(\bar y), \bar\phi(\bar y')] = \frac{\mathrm{i}\pi\beta^2}{2}s\left(\frac{\bar y-\bar y'}{R}\right) \tag{1.2}$$

where $\beta^2 > 0$ and $s(z)$ is the quasi-periodic extension of the sign function:

$$s(z) = 2n+1 \qquad n < z < n+1 \qquad s(n) = 2n \qquad n \in \mathbb{Z}. \tag{1.3}$$

In terms of a discretization of the Feigin–Fuks bosons, $\phi_m \equiv \phi\left(m\frac{R}{2N}\right)$, $\bar\phi_m \equiv \bar\phi\left(m\frac{R}{2N}\right)$, we defined the N-periodic operators $V_m^\pm$ living on a N-site lattice with spacing Δ and length $R = N\Delta$:

$$V_m^- \equiv \tfrac{1}{2}[(\phi_{2m-1} - \phi_{2m+1}) + (\phi_{2m-2} - \phi_{2m}) - (\bar\phi_{2m-1} - \bar\phi_{2m+1}) + (\bar\phi_{2m-2} - \bar\phi_{2m})] \tag{1.4}$$

$$V_m^+ \equiv \tfrac{1}{2}[(\bar\phi_{2m-1} - \bar\phi_{2m+1}) + (\bar\phi_{2m-2} - \bar\phi_{2m}) - (\phi_{2m-1} - \phi_{2m+1}) + (\phi_{2m-2} - \phi_{2m})]. \tag{1.5}$$

These operators are the quantum counterparts of the discretization of the mKdV variables v and $\bar v$ (1.1). The exponential operators $W_m^\pm \equiv \mathrm{e}^{\mathrm{i}V_m^\pm}$ satisfy, as a consequence of (1.2), the following exchange relations, first introduced in [15] (for $m = N$ the symbols $W_{N+1}^\pm$ should be read as $W_1^\pm$ respectively):

$$\begin{aligned} &W_{m+1}^\pm W_m^\pm = q^{\pm\frac{1}{2}} W_m^\pm W_{m+1}^\pm \qquad W_{m+1}^\pm W_m^\mp = q^{\mp\frac{1}{2}} W_m^\mp W_{m+1}^\pm \qquad W_m^+ W_m^- = q W_m^- W_m^+ \\ &[W_m^\sharp, W_n^{\sharp'}] = 0 \qquad \text{if} \quad 2 \leqslant |m-n| \leqslant N-2 \quad (\sharp, \sharp' = \pm). \end{aligned} \tag{1.6}$$

By using $W_m^\pm$ we constructed the *left* and *right* conformal monodromy matrices

$$M(\lambda) = L_N(\lambda)\ldots L_1(\lambda) \tag{1.7}$$

$$\bar M(\lambda) = \bar L_N(\lambda^{-1})\ldots \bar L_1(\lambda^{-1}) \tag{1.8}$$

together with the off-critical *right–left* and *left–right* monodromy matrices, depending on a *perturbation* parameter μ ($N \in 4\mathbb{N}$)

$$\begin{aligned} \boldsymbol{M}(\lambda) \equiv{}& \bar L_N(\mu^{\frac{1}{2}}\lambda^{-1})\bar L_{N-1}(\mu^{1/2}\lambda^{-1})L_{N-2}(\mu^{1/2}\lambda)L_{N-3}(\mu^{1/2}\lambda)\ldots \\ & \bar L_4(\mu^{1/2}\lambda^{-1})\bar L_3(\mu^{1/2}\lambda^{-1})L_2(\mu^{1/2}\lambda)L_1(\mu^{1/2}\lambda) \end{aligned} \tag{1.9}$$

$$\begin{aligned} \boldsymbol{M}'(\lambda) \equiv{}& L_N(\mu^{1/2}\lambda)L_{N-1}(\mu^{1/2}\lambda)\bar L_{N-2}(\mu^{1/2}\lambda^{-1})\bar L_{N-3}(\mu^{1/2}\lambda^{-1})\ldots \\ & L_4(\mu^{1/2}\lambda)L_3(\mu^{1/2}\lambda)\bar L_2(\mu^{1/2}\lambda^{-1})\bar L_1(\mu^{1/2}\lambda^{-1}). \end{aligned} \tag{1.10}$$

In these formulae left Lax operators $L_m(\lambda)$ and right Lax operators $\bar L_m(\lambda)$ are

$$L_m(\lambda) \equiv \begin{pmatrix} (W_m^-)^{-1} & \Delta\lambda W_m^+ \\ \Delta\lambda(W_m^+)^{-1} & W_m^- \end{pmatrix} \qquad \bar L_m(\lambda) \equiv \begin{pmatrix} (W_m^+)^{-1} & \Delta\lambda W_m^- \\ \Delta\lambda(W_m^-)^{-1} & W_m^+ \end{pmatrix}. \tag{1.11}$$

The monodromy matrix (1.7) and quantum Lax operator $L_m(\lambda)$ were first introduced in [6], starting from V_m, $\bar V_m$ and not from ϕ_n, $\bar\phi_n$. Quantum Lax operators $L_m(\lambda)$ and $\bar L_m(\lambda)$ are *non-ultralocal*, i.e. their operator entries on nearest-neighbouring sites do not commute. As a consequence, monodromy matrices (1.7)–(1.10) satisfy *braided* Yang–Baxter relations, first introduced in complete generality in [5, 6]. For instance we have that M (1.7) satisfies (a and b are two auxiliary spaces)

$$R_{ab}\left(\frac{\lambda}{\lambda'}\right) M_a(\lambda) Z_{ab}^{-1} M_b(\lambda') = M_b(\lambda') Z_{ba}^{-1} M_a(\lambda) R_{ab}\left(\frac{\lambda}{\lambda'}\right). \tag{1.12}$$

The matrix $\boldsymbol{M}'$ (1.10) satisfies the same relation, while the matrices $\bar{M}$ (1.8) and $\boldsymbol{M}$ (1.9) satisfy (1.12) with Z_{ab} replaced by Z_{ab}^{-1}. The usual numerical R-matrix is given by

$$R_{ab}(\xi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\xi^{-1}-\xi}{q^{-1}\xi^{-1}-q\xi} & \frac{q^{-1}-q}{q^{-1}\xi^{-1}-q\xi} & 0 \\ 0 & \frac{q^{-1}-q}{q^{-1}\xi^{-1}-q\xi} & \frac{\xi^{-1}-\xi}{q^{-1}\xi^{-1}-q\xi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad q = \mathrm{e}^{-\mathrm{i}\pi\beta^2} \tag{1.13}$$

and the numerical matrix $Z_{ab} = \mathrm{diag}(q^{-1/2}, q^{1/2}, q^{1/2}, q^{-1/2})$. In [7] we also showed that generators $W_m^{\pm}$ admit a realization in terms of *ultralocal* generators (canonical variables) x_m, p_m, $1 \leqslant m \leqslant N$, forming a position–momentum Heisenberg algebra

$$[x_m,\ x_n] = 0 \qquad [p_m,\ p_n] = 0 \qquad [x_m,\ p_n] = \frac{\mathrm{i}\pi\beta^2}{2}\delta_{m,n}. \tag{1.14}$$

Indeed, the operators

$$W_m^{\pm} = \mathrm{e}^{\mathrm{i}[\pm(x_{m+1}-x_m)-p_m]} \tag{1.15}$$

satisfy exchange relations (1.6). Of course, the symbol x_{N+1} appearing in (1.15) for $m = N$ should be read as x_1.

Using this realization we will establish a connection between the monodromy matrices (1.7)–(1.10) satisfying braided Yang–Baxter relations and some monodromy matrices satisfying the Yang–Baxter relation.

2. From the braided to the usual Yang–Baxter relation: conformal case

Let us rewrite the left Lax operator $L_m(\lambda)$ (1.11) using the realization (1.15) for $W_m^{\pm}$. We have that

$$L_m(\lambda) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}(x_m - x_{m+1} - p_m)} & \Delta\lambda \mathrm{e}^{-\mathrm{i}(x_m - x_{m+1} + p_m)} \\ \Delta\lambda \mathrm{e}^{\mathrm{i}(x_m - x_{m+1} + p_m)} & \mathrm{e}^{\mathrm{i}(x_m - x_{m+1} - p_m)} \end{pmatrix} = D_{m+1} U_m(\lambda) \tag{2.1}$$

where we have defined

$$D_m \equiv \begin{pmatrix} \mathrm{e}^{\mathrm{i}x_m} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}x_m} \end{pmatrix} \qquad U_m(\lambda) \equiv \begin{pmatrix} \mathrm{e}^{-\mathrm{i}(x_m - p_m)} & \Delta\lambda \mathrm{e}^{-\mathrm{i}(x_m + p_m)} \\ \Delta\lambda \mathrm{e}^{\mathrm{i}(x_m + p_m)} & \mathrm{e}^{\mathrm{i}(x_m - p_m)} \end{pmatrix}. \tag{2.2}$$

Of course, $D_{N+1} = D_1$ in (2.1) and the matrices D_m and $U_m(\lambda)$ depend only on ultralocal site variables. Therefore, we interpret formulae (2.1) as the decomposition of the Lax operator $L_m(\lambda)$ in its ultralocal components on the lattice sites.

Decomposition (2.1) implies the following form for the left monodromy matrix (1.7):

$$\begin{aligned} M(\lambda) &= L_N(\lambda) L_{N-1}(\lambda) \ldots L_2(\lambda) L_1(\lambda) \\ &= [D_1 U_N(\lambda)]\,[D_N U_{N-1}(\lambda)] \ldots [D_3 U_2(\lambda)]\,[D_2 U_1(\lambda)] \\ &= D_1\,[U_N(\lambda) D_N]\,[U_{N-1}(\lambda) D_{N-1}] \ldots [U_2(\lambda) D_2]\,[U_1(\lambda) D_1]\,D_1^{-1}. \end{aligned} \tag{2.3}$$

This means that the matrix

$$\tilde{M}(\lambda) \equiv D_1^{-1} M(\lambda) D_1 \tag{2.4}$$

can be written as

$$\tilde{M}(\lambda) = \tilde{L}_N(\lambda)\tilde{L}_{N-1}(\lambda)\ldots\tilde{L}_2(\lambda)\tilde{L}_1(\lambda) \tag{2.5}$$

in terms of the site *ultralocal* operators

$$\tilde{L}_m(\lambda) \equiv U_m(\lambda)D_m. \tag{2.6}$$

By using (2.2) in definition (2.6), we obtain the following expression for $\tilde{L}_m(\lambda)$:

$$\tilde{L}_m(\lambda) = q^{-1/4}\begin{pmatrix} e^{ip_m} & \Delta\lambda e^{-i(2x_m+p_m)} \\ \Delta\lambda e^{i(2x_m+p_m)} & e^{-ip_m} \end{pmatrix}. \tag{2.7}$$

From this expression and from commutation relations (1.14) it eventually follows that the ultralocal operators $\tilde{L}_m(\lambda)$ fulfil the Yang–Baxter relation

$$R_{ab}\left(\frac{\lambda}{\lambda'}\right)\tilde{L}_{am}(\lambda)\tilde{L}_{bm}(\lambda') = \tilde{L}_{bm}(\lambda')\tilde{L}_{am}(\lambda)R_{ab}\left(\frac{\lambda}{\lambda'}\right) \tag{2.8}$$

where the matrix R_{ab} is given by (1.13). As a consequence of the ultralocality of $\tilde{L}_m$, we obtain as usual that the matrix $\tilde{M}$, in its turn, satisfies the Yang–Baxter relation as well:

$$R_{ab}\left(\frac{\lambda}{\lambda'}\right)\tilde{M}_a(\lambda)\tilde{M}_b(\lambda') = \tilde{M}_b(\lambda')\tilde{M}_a(\lambda)R_{ab}\left(\frac{\lambda}{\lambda'}\right). \tag{2.9}$$

Therefore, the transfer matrix $\tilde{\tau}(\lambda) \equiv \mathrm{Tr}\,\tilde{M}(\lambda)$ commutes with itself for different values of the spectral parameter:

$$[\tilde{\tau}(\lambda), \tilde{\tau}(\lambda')] = 0 \tag{2.10}$$

and, as a consequence, generates infinite conserved charges in involution.

Observation. The aforementioned results of this section have been obtained using the realization (1.15). However, the transformation of an algebra defined by the braided Yang–Baxter relation into another algebra defined by the usual Yang–Baxter relation is more general, as follows from the following theorem.

Theorem 1. *Let M be a matrix satisfying the braided Yang–Baxter relation (1.12). If there exists an invertible matrix D satisfying the conditions*

$$M_a(\lambda) = D_bM_a(\lambda)D_b^{-1}Z_{ab} \qquad [D_aD_b, R_{ab}(\xi)] = 0 \qquad [D_a, D_b] = 0 \tag{2.11}$$

and if R_{ab} and Z_{ab} entering (1.12) satisfy

$$Z_{ab}R_{ab}(\xi) = R_{ab}(\xi)Z_{ba} \tag{2.12}$$

then $\tilde{M}(\lambda) = D^{-1}M(\lambda)D$ satisfies the Yang–Baxter relation (2.9).

The proof comes from direct calculations. In the algebraic context the meaning of the theorem is that the algebra generated by the entries of a matrix M satisfying the braided Yang–Baxter relation (1.12), with R_{ab} and Z_{ab} obeying (2.12), is isomorphic to the algebra generated by the entries of the *unbraided* matrix $\tilde{M}(\lambda) = D^{-1}M(\lambda)D$ satisfying the Yang–Baxter relation (2.9), if a matrix (of formal elements) D obeying conditions (2.11) exists: an *unbraiding* isomorphism. In the (m)KdV theory (2.12) is evidently true and in the realization (1.15) conditions (2.11) are satisfied by the matrix D_1, defined in (2.2). A still open problem concerns the possibility of choosing D in such a way that the above isomorphism extends to the respective Hopf algebras. Let us finally remark that the most general definition of braided Yang–Baxter algebras [5] uses two numerical matrices, Z_{ab} and $\tilde{Z}_{ab}$: the defining relation is $R_{ab}\tilde{Z}_{ba}^{-1}M_aZ_{ab}^{-1}M_b = \tilde{Z}_{ab}^{-1}M_bZ_{ba}^{-1}M_aR_{ab}$. However, the aforementioned isomorphism into a Yang–Baxter algebra exists when condition (2.12) holds for R_{ab} and $\tilde{Z}_{ab}$ and this reduces the above general defining relation to (1.12).

Let us now study in detail the invertible map from the left conformal monodromy matrix M into the unbraided monodromy matrix $\tilde{M}$. From relation (2.4) and from the definition of D_1 (2.2) it follows that the relations between the matrix elements of M and $\tilde{M}$:

$$M(\lambda) \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \qquad \tilde{M}(\lambda) \equiv \begin{pmatrix} \tilde{A}(\lambda) & \tilde{B}(\lambda) \\ \tilde{C}(\lambda) & \tilde{D}(\lambda) \end{pmatrix} \tag{2.13}$$

are the following:

$$\begin{aligned} A(\lambda) &= \mathrm{e}^{\mathrm{i}x_1}\tilde{A}(\lambda)\mathrm{e}^{-\mathrm{i}x_1} \qquad B(\lambda) = \mathrm{e}^{\mathrm{i}x_1}\tilde{B}(\lambda)\mathrm{e}^{\mathrm{i}x_1} \\ C(\lambda) &= \mathrm{e}^{-\mathrm{i}x_1}\tilde{C}(\lambda)\mathrm{e}^{-\mathrm{i}x_1} \qquad D(\lambda) = \mathrm{e}^{-\mathrm{i}x_1}\tilde{D}(\lambda)\mathrm{e}^{\mathrm{i}x_1}. \end{aligned} \tag{2.14}$$

In order to simplify relations (2.14) we rewrite the matrix $\tilde{M}$ as follows:

$$\begin{aligned} \tilde{M}(\lambda) &= \tilde{L}_N(\lambda)\tilde{L}_{N-1}(\lambda)\ldots\tilde{L}_2(\lambda)\tilde{L}_1(\lambda) \\ &= \left[\tilde{L}_N(\lambda)\tilde{L}_{N-1}(\lambda)\ldots\tilde{L}_2(\lambda)\right]\tilde{L}_1(\lambda) \equiv \tilde{M}'(\lambda)\tilde{L}_1(\lambda). \end{aligned} \tag{2.15}$$

Hence, from (2.14), (2.7) and (2.15) we obtain, with obvious notations:

$$A(\lambda) = \mathrm{e}^{\mathrm{i}x_1}\left[\tilde{A}'(\lambda)\mathrm{e}^{\mathrm{i}p_1} + \tilde{B}'(\lambda)\Delta\lambda\mathrm{e}^{\mathrm{i}(2x_1+p_1)}\right]\mathrm{e}^{-\mathrm{i}x_1}q^{-1/4}. \tag{2.16}$$

The factor $\mathrm{e}^{-\mathrm{i}x_1}$ commutes with $\tilde{A}'(\lambda)$ and $\tilde{B}'(\lambda)$ because these depend only on the ultralocal variables of the sites $2, \ldots, N$. The exchange with the other factors in the square bracket is regulated by (1.14) and produces

$$A(\lambda) = q^{1/2}\tilde{A}(\lambda). \tag{2.17}$$

The same procedure applied to the other elements of M and $\tilde{M}$ gives

$$D(\lambda) = q^{1/2}\tilde{D}(\lambda) \tag{2.18}$$

$$B(\lambda) = q^{1/2}\mathrm{e}^{2\mathrm{i}x_1}\tilde{B}(\lambda) \qquad C(\lambda) = q^{1/2}\mathrm{e}^{-2\mathrm{i}x_1}\tilde{C}(\lambda). \tag{2.19}$$

Let us finally remark that from (2.17) and (2.18) it follows that

$$\tilde{\tau}(\lambda) = \tilde{A}(\lambda) + \tilde{D}(\lambda) = q^{-1/2}[A(\lambda) + D(\lambda)] = q^{-1/2}\tau(\lambda) \tag{2.20}$$

i.e. the transfer matrix $\tilde{\tau}(\lambda) = \mathrm{Tr}\,\tilde{M}(\lambda)$ is *proportional* to the transfer matrix $\tau(\lambda) \equiv \mathrm{Tr}\,M(\lambda)$. Hence $\tilde{\tau}$ describes the same observables as τ, with the advantage, however, of coming from a monodromy matrix made up of ultralocal site operators. Since $B, \tilde{B}$ and $C, \tilde{C}$ are respectively different, the diagonalizations of τ and $\tilde{\tau}$ by means of algebraic Bethe ansatz are made, in principle, on different (but related) vector spaces. In [7] we diagonalized τ and conjectured that it describes, in the continuum limit, the left sector of minimal CFTs on a cylinder. In a forthcoming publication [16] we will write the Bethe equations and the eigenvalues/eigenvectors of the transfer matrix $\tilde{\tau}$ and we will show that they coincide with the homonymous quantities in the Liouville model [17]. We will also disentangle the comparison between the eigenvectors of τ and $\tilde{\tau}$.

We are going to repeat the *unbraiding* procedure, making use of the right Lax operators $\bar{L}_m(\lambda)$ (1.11). At first, we write $\bar{L}_m(\lambda)$ in terms of ultralocal operators. Using (1.15) we have that

$$\bar{L}_m(\lambda) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(x_m - x_{m+1} + p_m)} & \Delta\lambda\mathrm{e}^{\mathrm{i}(x_m - x_{m+1} - p_m)} \\ \Delta\lambda\mathrm{e}^{-\mathrm{i}(x_m - x_{m+1} - p_m)} & \mathrm{e}^{-\mathrm{i}(x_m - x_{m+1} + p_m)} \end{pmatrix} = D_{m+1}^{-1}\bar{U}_m(\lambda) \tag{2.21}$$

where D_m is still given by (2.2) and

$$\bar{U}_m(\lambda) \equiv \begin{pmatrix} \mathrm{e}^{\mathrm{i}(x_m + p_m)} & \Delta\lambda\mathrm{e}^{\mathrm{i}(x_m - p_m)} \\ \Delta\lambda\mathrm{e}^{-\mathrm{i}(x_m - p_m)} & \mathrm{e}^{-\mathrm{i}(x_m + p_m)} \end{pmatrix}. \tag{2.22}$$

It is evident that $\bar{U}_m(\lambda)$ depends only on ultralocal site variables and hence that formula (2.21) decomposes the right Lax operator into its ultralocal components. Decomposition (2.21) allows us to rewrite the right monodromy matrix (1.8) as follows:

$$\begin{aligned}\bar{M}(\lambda) &= \bar{L}_N(\lambda^{-1})\bar{L}_{N-1}(\lambda^{-1})\ldots\bar{L}_2(\lambda^{-1})\bar{L}_1(\lambda^{-1})\\ &= [D_1^{-1}\bar{U}_N(\lambda^{-1})]\,[D_N^{-1}\bar{U}_{N-1}(\lambda^{-1})]\ldots[D_3^{-1}\bar{U}_2(\lambda^{-1})]\,[D_2^{-1}\bar{U}_1(\lambda^{-1})]\\ &= D_1^{-1}\,[\bar{U}_N(\lambda^{-1})D_N^{-1}]\,[\bar{U}_{N-1}(\lambda^{-1})D_{N-1}^{-1}]\ldots[\bar{U}_2(\lambda^{-1})D_2^{-1}]\,[\bar{U}_1(\lambda^{-1})D_1^{-1}]\,D_1.\end{aligned}$$

This means that the matrix

$$\tilde{\bar{M}}(\lambda) \equiv D_1\bar{M}(\lambda)D_1^{-1} \tag{2.23}$$

can be written as

$$\tilde{\bar{M}}(\lambda) = \tilde{\bar{L}}_N(\lambda^{-1})\tilde{\bar{L}}_{N-1}(\lambda^{-1})\ldots\tilde{\bar{L}}_2(\lambda^{-1})\tilde{\bar{L}}_1(\lambda^{-1}) \tag{2.24}$$

in terms of the ultralocal operators

$$\tilde{\bar{L}}_m(\lambda) \equiv \bar{U}_m(\lambda)D_m^{-1} = q^{1/4}\begin{pmatrix} \mathrm{e}^{\mathrm{i}p_m} & \Delta\lambda\mathrm{e}^{\mathrm{i}(2x_m-p_m)} \\ \Delta\lambda\mathrm{e}^{-\mathrm{i}(2x_m-p_m)} & \mathrm{e}^{-\mathrm{i}p_m}\end{pmatrix}. \tag{2.25}$$

From this realization of the operators $\tilde{\bar{L}}_m(\lambda)$ and from commutation relations (1.14) the Yang–Baxter relation follows:

$$R_{ab}\left(\frac{\lambda}{\lambda'}\right)\tilde{\bar{L}}_{am}\left(\frac{1}{\lambda}\right)\tilde{\bar{L}}_{bm}\left(\frac{1}{\lambda'}\right) = \tilde{\bar{L}}_{bm}\left(\frac{1}{\lambda'}\right)\tilde{\bar{L}}_{am}\left(\frac{1}{\lambda}\right)R_{ab}\left(\frac{\lambda}{\lambda'}\right). \tag{2.26}$$

Since $\tilde{\bar{L}}_m$ are ultralocal, the Yang–Baxter relation is also true for the matrix $\tilde{\bar{M}}$:

$$R_{ab}\left(\frac{\lambda}{\lambda'}\right)\tilde{\bar{M}}_a(\lambda)\tilde{\bar{M}}_b(\lambda') = \tilde{\bar{M}}_b(\lambda')\tilde{\bar{M}}_a(\lambda)R_{ab}\left(\frac{\lambda}{\lambda'}\right) \tag{2.27}$$

and gives rise to the commutativity of the transfer matrix $\tilde{\bar{\tau}}(\lambda) \equiv \mathrm{Tr}\,\tilde{\bar{M}}(\lambda)$ with itself for different values of the spectral parameter:

$$[\tilde{\bar{\tau}}(\lambda), \tilde{\bar{\tau}}(\lambda')] = 0. \tag{2.28}$$

The unbraided monodromy matrix $\tilde{\bar{M}}$ is directly connected to the right conformal monodromy matrix $\bar{M}$ by (2.23). More explicitly, the matrix elements of $\bar{M}$ and $\tilde{\bar{M}}$:

$$\bar{M}(\lambda) \equiv \begin{pmatrix}\bar{A}(\lambda) & \bar{B}(\lambda)\\ \bar{C}(\lambda) & \bar{D}(\lambda)\end{pmatrix} \qquad \tilde{\bar{M}}(\lambda) \equiv \begin{pmatrix}\tilde{\bar{A}}(\lambda) & \tilde{\bar{B}}(\lambda)\\ \tilde{\bar{C}}(\lambda) & \tilde{\bar{D}}(\lambda)\end{pmatrix} \tag{2.29}$$

are related in this way:

$$\begin{aligned}&\bar{A}(\lambda) = \mathrm{e}^{-\mathrm{i}x_1}\tilde{\bar{A}}(\lambda)\mathrm{e}^{\mathrm{i}x_1} \qquad && \bar{B}(\lambda) = \mathrm{e}^{-\mathrm{i}x_1}\tilde{\bar{B}}(\lambda)\mathrm{e}^{-\mathrm{i}x_1}\\ &\bar{C}(\lambda) = \mathrm{e}^{\mathrm{i}x_1}\tilde{\bar{C}}(\lambda)\mathrm{e}^{\mathrm{i}x_1} && \bar{D}(\lambda) = \mathrm{e}^{\mathrm{i}x_1}\tilde{\bar{D}}(\lambda)\mathrm{e}^{-\mathrm{i}x_1}.\end{aligned} \tag{2.30}$$

Using the same technique as before—equations (2.15) and (2.16)—we simplify (2.30) to

$$\bar{A}(\lambda) = q^{-1/2}\tilde{\bar{A}}(\lambda) \qquad \bar{D}(\lambda) = q^{-1/2}\tilde{\bar{D}}(\lambda) \tag{2.31}$$

$$\bar{B}(\lambda) = q^{-1/2}\mathrm{e}^{-2\mathrm{i}x_1}\tilde{\bar{B}}(\lambda) \qquad \bar{C}(\lambda) = q^{-1/2}\mathrm{e}^{2\mathrm{i}x_1}\tilde{\bar{C}}(\lambda). \tag{2.32}$$

In conclusion, we have that

$$\tilde{\bar{\tau}}(\lambda) = \tilde{\bar{A}}(\lambda) + \tilde{\bar{D}}(\lambda) = q^{1/2}[\bar{A}(\lambda) + \bar{D}(\lambda)] = q^{1/2}\bar{\tau}(\lambda) \tag{2.33}$$

i.e. the transfer matrix $\tilde{\bar{\tau}}(\lambda) = \mathrm{Tr}\,\tilde{\bar{M}}(\lambda)$ is proportional to the transfer matrix $\bar{\tau}(\lambda) \equiv \mathrm{Tr}\,\bar{M}(\lambda)$ and hence describes the same observables. We can comment on these results concerning the right sector in the same way we have done for those about the left one, simply turning the word 'left' into 'right'.

3. From the braided to the usual Yang–Baxter relation: off-critical case

We now want to extend our unbraiding procedure by connecting the off-critical monodromy matrix $\boldsymbol{M}$ (1.9) with a suitable monodromy matrix satisfying the Yang–Baxter relation. Following what we have just done in the conformal case, we want to write (1.9) in terms of ultralocal operators. Decomposition (2.1) for L_m and decomposition (2.21) for $\bar{L}_m$ give

$$\begin{aligned}
\boldsymbol{M}(\lambda) &= [D_1^{-1}\bar{U}_N(\mu^{1/2}\lambda^{-1})]\,[D_N^{-1}\bar{U}_{N-1}(\mu^{1/2}\lambda^{-1})][D_{N-1}U_{N-2}(\mu^{1/2}\lambda)]\\
&\quad\times[D_{N-2}U_{N-3}(\mu^{1/2}\lambda)]\cdots[D_5^{-1}\bar{U}_4(\mu^{1/2}\lambda^{-1})]\,[D_4^{-1}\bar{U}_3(\mu^{1/2}\lambda^{-1})]\\
&\quad\times[D_3U_2(\mu^{1/2}\lambda)]\,[D_2U_1(\mu^{1/2}\lambda)]\\
&= D_1^{-1}[\bar{U}_N(\mu^{1/2}\lambda^{-1})D_N^{-1}]\,[\bar{U}_{N-1}(\mu^{1/2}\lambda^{-1})D_{N-1}][U_{N-2}(\mu^{1/2}\lambda)D_{N-2}]\\
&\quad\times[U_{N-3}(\mu^{1/2}\lambda)D_{N-3}^{-1}]\cdots[\bar{U}_4(\mu^{1/2}\lambda^{-1})D_4^{-1}]\,[\bar{U}_3(\mu^{1/2}\lambda^{-1})D_3]\\
&\quad\times[U_2(\mu^{1/2}\lambda)D_2]\,[U_1(\mu^{1/2}\lambda)D_1^{-1}]D_1.
\end{aligned}\tag{3.1}$$

This means that the matrix

$$\tilde{\boldsymbol{M}}(\lambda) \equiv D_1\,\boldsymbol{M}(\lambda)\,D_1^{-1} \tag{3.2}$$

can be written as

$$\tilde{\boldsymbol{M}}(\lambda) = \overset{\longleftarrow}{\prod_{i=1}^{N/4}} \tilde{\bar{L}}_{4i}(\mu^{1/2}\lambda^{-1})\tilde{\bar{L}}'_{4i-1}(\mu^{1/2}\lambda^{-1})\tilde{L}_{4i-2}(\mu^{1/2}\lambda)\tilde{L}'_{4i-3}(\mu^{1/2}\lambda) \tag{3.3}$$

in terms of the ultralocal operators (2.6) and (2.25) and

$$\tilde{\bar{L}}'_{4i-1}(\lambda) \equiv \bar{U}_{4i-1}(\lambda)D_{4i-1} \tag{3.4}$$

$$\tilde{L}'_{4i-3}(\lambda) \equiv U_{4i-3}(\lambda)D_{4i-3}^{-1}. \tag{3.5}$$

The new operators (3.4) and (3.5) inherit the realization

$$\tilde{\bar{L}}'_m(\lambda) = q^{-1/4}\begin{pmatrix} \mathrm{e}^{\mathrm{i}(2x_m+p_m)} & \Delta\lambda\mathrm{e}^{-\mathrm{i}p_m}\\ \Delta\lambda\mathrm{e}^{\mathrm{i}p_m} & \mathrm{e}^{-\mathrm{i}(2x_m+p_m)}\end{pmatrix} \tag{3.6}$$

$$\tilde{L}'_m(\lambda) = q^{1/4}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}(2x_m-p_m)} & \Delta\lambda\mathrm{e}^{-\mathrm{i}p_m}\\ \Delta\lambda\mathrm{e}^{\mathrm{i}p_m} & \mathrm{e}^{\mathrm{i}(2x_m-p_m)}\end{pmatrix}. \tag{3.7}$$

We already know—formulae (2.8) and (2.26)—that operators $\tilde{L}_m(\lambda)$ and $\tilde{\bar{L}}_m(\lambda^{-1})$ satisfy the Yang–Baxter relation and this is also true for operators $\tilde{L}'_m(\lambda)$ and $\tilde{\bar{L}}'_m(\lambda^{-1})$, from direct calculation which uses (3.6) and (3.7). Since all these operators are ultralocal, the unbraided matrix $\tilde{\boldsymbol{M}}(\lambda)$ also satisfies the Yang–Baxter relation

$$R_{ab}\left(\frac{\lambda}{\lambda'}\right)\tilde{\boldsymbol{M}}_a(\lambda)\tilde{\boldsymbol{M}}_b(\lambda') = \tilde{\boldsymbol{M}}_b(\lambda')\tilde{\boldsymbol{M}}_a(\lambda)R_{ab}\left(\frac{\lambda}{\lambda'}\right) \tag{3.8}$$

and produces a transfer matrix $\tilde{t}(\lambda) \equiv \mathrm{Tr}\,\tilde{\boldsymbol{M}}(\lambda)$ commuting with itself for different values of the spectral parameter:

$$[\tilde{t}(\lambda), \tilde{t}(\lambda')] = 0. \tag{3.9}$$

Now, we analyse in more detail the relation between the off-critical monodromy matrix $\boldsymbol{M}$ and the monodromy matrix $\tilde{\boldsymbol{M}}$. From formula (3.2) it follows that this relation formally coincides with relation (2.23) between $\bar{M}$ and $\tilde{\bar{M}}$. Therefore, we need to rewrite properly (2.30) in the form:

$$\boldsymbol{A}(\lambda) = \mathrm{e}^{-\mathrm{i}x_1}\tilde{\boldsymbol{A}}(\lambda)\mathrm{e}^{\mathrm{i}x_1} \qquad \boldsymbol{B}(\lambda) = \mathrm{e}^{-\mathrm{i}x_1}\tilde{\boldsymbol{B}}(\lambda)\mathrm{e}^{-\mathrm{i}x_1} \tag{3.10}$$

$$\boldsymbol{C}(\lambda) = \mathrm{e}^{\mathrm{i}x_1}\tilde{\boldsymbol{C}}(\lambda)\mathrm{e}^{\mathrm{i}x_1} \qquad \boldsymbol{D}(\lambda) = \mathrm{e}^{\mathrm{i}x_1}\tilde{\boldsymbol{D}}(\lambda)\mathrm{e}^{-\mathrm{i}x_1}. \tag{3.11}$$

Besides, as in that case we can exchange the factors $e^{\pm i x_1}$ with the elements $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ of $\tilde{M}$:

$$A(\lambda) = q^{-1/2}\tilde{A}(\lambda) \qquad D(\lambda) = q^{-1/2}\tilde{D}(\lambda) \tag{3.12}$$

$$B(\lambda) = q^{-1/2}e^{-2ix_1}\tilde{B}(\lambda) \qquad C(\lambda) = q^{-1/2}e^{2ix_1}\tilde{C}(\lambda). \tag{3.13}$$

As in the conformal case, we remark that $t(\lambda) \equiv \operatorname{Tr} M(\lambda)$ and $\tilde{t}(\lambda) = \operatorname{Tr}\tilde{M}(\lambda)$ are proportional:

$$\tilde{t}(\lambda) = q^{1/2}t(\lambda). \tag{3.14}$$

Hence they describe the same observables. Likewise, since B, $\tilde{B}$ and C, $\tilde{C}$ are different, the diagonalization of t and $\tilde{t}$ by means of Bethe ansatz is made, in principle, on different (but related) vector spaces. In [7] we diagonalized t and conjectured that it describes, in the cylinder continuum limit, minimal CFTs perturbed by the $\Phi_{1,3}$ primary operator. In a forthcoming publication [16] we will write the Bethe equations and the eigenvalues/eigenvectors of the transfer matrix $\tilde{t}$ and we will show that they coincide with the Bethe equations and the eigenvalues/eigenvectors of the transfer matrix for the lattice sine-Gordon model [18]. Since minimal CFTs perturbed by $\Phi_{1,3}$ are a restriction of the sine-Gordon model [13], the Bethe ansatz construction of the eigenvectors of t and $\tilde{t}$ will be useful to prove our conjecture.

Eventually, we give briefly analogous results for the other off-critical monodromy matrix (1.10) with entries defined by

$$M'(\lambda) \equiv \begin{pmatrix} A'(\lambda) & B'(\lambda) \\ C'(\lambda) & D'(\lambda) \end{pmatrix}. \tag{3.15}$$

Through the same unbraiding procedure we end up with a monodromy matrix built up by ultralocal site operators:

$$\tilde{M}'(\lambda) \equiv D_1^{-1}M'(\lambda)D_1 \tag{3.16}$$

which can be written as follows:

$$\tilde{M}'(\lambda) = \overleftarrow{\prod_{i=1}^{N/4}} \tilde{L}_{4i}(\mu^{1/2}\lambda)\tilde{L}'_{4i-1}(\mu^{1/2}\lambda)\tilde{\tilde{L}}_{4i-2}(\mu^{1/2}\lambda^{-1})\tilde{\tilde{L}}'_{4i-3}(\mu^{1/2}\lambda^{-1}). \tag{3.17}$$

Likewise, the matrix $\tilde{M}'(\lambda)$ satisfies the Yang–Baxter relation and the connection between its entries and those of the monodromy matrix $M'(\lambda)$ can be written down explicitly:

$$A'(\lambda) = q^{1/2}\tilde{A}'(\lambda) \qquad D'(\lambda) = q^{1/2}\tilde{D}'(\lambda) \tag{3.18}$$

$$B'(\lambda) = q^{1/2}e^{2ix_1}\tilde{B}'(\lambda) \qquad C'(\lambda) = q^{1/2}e^{-2ix_1}\tilde{C}'(\lambda). \tag{3.19}$$

Therefore, $t'(\lambda) \equiv \operatorname{Tr} M'(\lambda)$ and $\tilde{t}'(\lambda) \equiv \operatorname{Tr}\tilde{M}'(\lambda)$ are proportional,

$$\tilde{t}'(\lambda) = q^{-1/2}t'(\lambda) \tag{3.20}$$

and consequently describe the same observables. In a forthcoming publication [16] we will write the Bethe equations and the eigenvalues/eigenvectors of $\tilde{t}'$ and we will show that they coincide with the homonymous quantities for $\tilde{t}$ and hence with the homonymous quantities for the lattice sine-Gordon model [18].

Observation. It is important to remark that the existence of the ultralocal Lax operators (2.7) and (2.25), satisfying the Yang–Baxter relations (2.8) and (2.26), allows us to define, if N is even, the *factorized* monodromy matrix

$$\begin{aligned} M^F(\lambda) = {} & \tilde{L}_N(\mu^{1/2}\lambda)\tilde{L}_{N-2}(\mu^{1/2}\lambda)\ldots\tilde{L}_4(\mu^{1/2}\lambda)\tilde{L}_2(\mu^{1/2}\lambda) \\ & \times\tilde{\tilde{L}}_{N-1}(\mu^{1/2}\lambda^{-1})\tilde{\tilde{L}}_{N-3}(\mu^{1/2}\lambda^{-1})\ldots\tilde{\tilde{L}}_3(\mu^{1/2}\lambda^{-1})\tilde{\tilde{L}}_1(\mu^{1/2}\lambda^{-1}) \end{aligned} \tag{3.21}$$

which satisfies the Yang–Baxter relation:

$$R_{ab}\left(\frac{\lambda}{\lambda'}\right) M_a^F(\lambda) M_b^F(\lambda') = M_b^F(\lambda') M_a^F(\lambda) R_{ab}\left(\frac{\lambda}{\lambda'}\right). \tag{3.22}$$

The matrix (3.21) is made up of two parts. The left part contains only operators $\tilde{L}_{2m}$ on even sites, the right part only operators $\tilde{\bar{L}}_{2m+1}$ on odd sites. From formulae (2.1), (2.6), (2.21) and (2.25) it follows that $\tilde{L}_{2m} = D_{2m+1}^{-1} L_{2m} D_{2m}$ and that $\tilde{\bar{L}}_{2m+1} = D_{2m+2} \bar{L}_{2m+1} D_{2m+1}^{-1}$, where D_m, given by (2.2), is a simple diagonal matrix. Since in the scaling limit left and right Lax operators L_{2m} and $\bar{L}_{2m+1}$ become completely chiral and antichiral respectively (see section 9 of [7]), in the same limit matrix (3.21) is the product of two matrices, one depending mainly on the Feigin–Fuks boson ϕ, the other depending mainly on $\bar{\phi}$. These matrices, however, are not completely chiral, because of the presence of the diagonal matrices D_m inside them. Nevertheless, directly in the scaling limit ($\Delta \to 0$) a chiral part by anti–chiral part factorized monodromy matrix was also discovered in [9] to satisfy the Yang–Baxter relation. Apparently, that matrix is slightly different from the scaling limit of (3.21) and from the lattice we did not find, now, any way to reproduce exactly that matrix, preserving the Yang–Baxter relation. Therefore, in order to compare our factorized monodromy matrix with that in [9], we need to diagonalize by means of algebraic Bethe ansatz the transfer matrix $t^F(\lambda) \equiv \mathrm{Tr}\, M^F(\lambda)$, which, as a consequence of (3.22), commutes for different values of the spectral parameter. This will be the issue of a forthcoming publication [16].

4. Perspectives

We have found a Yang–Baxter relation in theories controlled by a braided Yang–Baxter algebra, working out the construction in physical examples involving the lattice (m)KdV theory as the main ingredient. Since this theory is the prototype of non-ultralocal theories and we have gone from non-ultralocal commutators to ultralocal ones, our treatment is completely general for what concerns lattice theories (and their continuum limit). For instance, our method could be applied to the quantum theory described in [19]—which exhausts all the other integrable perturbations of minimal CFTs—or to W_3 symmetric CFTs [20] as described in [21]. Moreover, this unbraiding transformation has also been formulated under a general point of view, giving rise to an algebra isomorphism provided a matrix D, satisfying suitable conditions, exists. In the next future the proposed unbraided monodromy matrices—especially the factorized monodromy matrix—will be worthy of being investigated. Indeed, we will better understand the spectrum of braided and unbraided theories by comparing the algebraic Bethe ansatz representations [16]. In this way it will also be possible to perform the (cylinder) continuum limit and obtain the spectrum of field theories (in finite volume, i.e. at finite temperature).

DF is grateful to F Colomo, E Ercolessi and M Stanishkov for discussions and thanks EPRSC for grant GR/M66370. MR thanks EPRSC for grant GR/M97497. This work has been partially supported by EC TMR contract ERBFMRXCT960012.

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